

Note 1: Answers must be fully explained (unless otherwise specified). If the answer is not fully completed, you should explain e.g. some special cases you have worked out, whatever reasons you have for coming up with any stated conjectures, whatever partial progress you have made (e.g. if you have solved what you believe to be an intermediate result).

Allowed activities on Midterm are found in the table at the end of this document:

Any additional clarification/corrections to questions will be posted as an announcement on Canvas.

Note 2: We use both $\{a_i\}$, and $\{a_i\}_{i=1}^\infty$ to denote a sequence of numbers indexed by $i \in \mathbb{N} \setminus \{0\}$. Questions involving limits of series should be approached using the $\epsilon - N$ definition of limits.

- (1) Let X, Y be two sets with equivalence relations \sim_X and \sim_Y respectively. Consider the two binary relations on $X \times Y$
 - $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 \sim_X x_2$ and $y_1 \sim_Y y_2$.
 - $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 \sim_X x_2$ or $y_1 \sim_Y y_2$.
 - (a) For each of these binary relations, either prove or disprove that it is an equivalence relation²
 - (b) We now let \sim denote the binary relation above that was found to be an equivalence relation. Find a bijection

$$(X \times Y) / \sim \rightarrow (X / \sim_X) \times (Y / \sim_Y).$$

- (2) Suppose that X is a finite set with an equivalence relation \sim . Suppose that $f : X \rightarrow X$ descends to a function $g : X / \sim \rightarrow X / \sim$ (ie. g is specified by $g([x]) = [f(x)]$ and is a function). Prove or disprove that for any such X, f, \sim_X :
 - (a) If f has an inverse then g has an inverse.
 - (b) If g has an inverse then f has an inverse.
- (3) Does answer to Q2 change if we remove the finiteness assumption?
- (4) Do the following sequences converge? If they converge what do they converge to?

Use the $\epsilon - N$ definition of convergence, or results proven in class.

 - (a) $\{\frac{i^2}{i+1} - \frac{i^2-i}{i-1}\}$.
 - (b)

$$\left\{ \frac{\cos(\pi i + \frac{1}{i})}{\cos(\frac{\pi i}{2} - \frac{1}{i})} \right\}.$$

- (5) Prove or disprove: Let $\{x_i\}$ be a sequence of real numbers. Suppose that the sequences $\{x_{2i}\}$, $\{x_{2i+1}\}$, and $\{x_{i^4}\}$ are convergent. Then the sequence $\{x_i\}$ is convergent.

Clarification of notation: Consider the sequence given by $x_i = i$, then the sequence is x_{2i} is the sequence $\{2, 4, 6, 8, \dots\}$, and the sequence x_{i^2} is the sequence $\{1, 4, 9, 16, \dots\}$.
- (6) Suppose that $\{a_i\}$ is a convergent sequence of real numbers with limit L (a real number), such that $a_i \neq 0$ for any $i \in \mathbb{N} \setminus \{0\}$. For which values of L does it follow that the sequence

$$\left\{ \frac{a_i}{a_{i+1}} \right\}$$

converges? For these values of L what does the sequence converge to?

¹It is of course easy to inadvertently find solutions when looking at resources on the topic. Do not be concerned about this, but the point is that you should be working out solutions to the problems not finding them.

²For *any* sets and equivalence relations X, Y, \sim_X, \sim_Y .

Note: For the values of L such that it does not follow that this sequence converges you must explain why this does not follow.

Activity	Midterm 2
Discussing Problems with other people	Allowed
Discussing Questions on Ed Discussion	Allowed
Asking Me for Clarification (by email)	Allowed
Look at final written solutions of other students prior to submission.	Not Allowed
Post Problem Online	Not Allowed
Use any Textbook	Allowed
Look up concepts online	Allowed
Search for solution to specific Problem online ¹	Not Allowed

TABLE 1. Allowed Activities on Midterm 2